ON THE DDVV CONJECTURE AND THE COMASS IN CALIBRATED GEOMETRY (II)

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1. Introduction

Let M^n be an n dimensional manifold isometrically immersed into the space form $N^{n+m}(c)$ of constant sectional curvature c. Define the normalized scalar curvature ρ and $\rho^\perp$ for the tangent bundle and the normal bundle as follows:

$$\begin{aligned}
\rho &= \frac{2}{n(n-1)}\sum_{1=i<j}^{n} R(e_i,e_j,e_j,e_i),\\
\rho^\perp &= \frac{2}{n(n-1)}\left(\sum_{1=i<j}^{n}\sum_{1=r<s}^{m}\langle R^\perp(e_i,e_j)\xi_r,\xi_s\rangle^2\right)^{\frac{1}{2}},
\end{aligned}\tag{1}$$

where $\{e_1,\cdots,e_n\}$ (resp. $\{\xi_1,\cdots,\xi_m\}$) is an orthonormal basis of the tangent (resp. normal space) at the point $x\in M$, and $R,R^\perp$ are the curvature tensors for the tangent and normal bundles, respectively.

In the study of submanifold theory, De Smet, Dillen, Verstraelen, and Vrancken [5] made the following *DDVV Conjecture*:

Conjecture 1. *Let h be the second fundamental form, and let $H=\frac{1}{n}\operatorname{trace} h$ be the mean curvature tensor. Then*

$$\rho+\rho^\perp\le |H|^2+c.$$

A weaker version of the above conjecture,

$$\rho\le |H|^2+c,$$

was proved in [2]. An alternate proof is in [7].

In [5], the authors proved the following

Theorem 1. *If $m=2$, then the conjecture is true.*

In this paper, we prove the conjecture in the case $m=3$. In the next version of this paper, we will prove $P(n,m)$.

This paper is the continuation of the previous paper [4], where the case $n=3$ was proved.

Let $x\in M$ be a fixed point and let (h^r_{ij}) $(i,j=1,\cdots,n$ and $r=1,\cdots,m)$ be the coefficients of the second fundamental form under some orthonormal basis. Then by Suceavă [8], or [6],

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Conjecture 1 can be formulated as an inequality with respect to the coefficients h_{ij}^r as follows:

$$\begin{aligned}&\sum_{r=1}^{m}\sum_{1=i<j}^{n}(h_{ii}^r-h_{jj}^r)^2+2n\sum_{r=1}^{m}\sum_{1=i<j}^{n}(h_{ij}^r)^2\\ &\geq 2n\left(\sum_{1=r<s}^{m}\sum_{1=i<j}^{n}\left(\sum_{k=1}^{n}(h_{ik}^r h_{jk}^s-h_{ik}^s h_{jk}^r)\right)^2\right)^{\frac{1}{2}}.\end{aligned}\tag{2}$$

Suppose that $A_1, A_2, \cdots, A_m$ are $n\times n$ symmetric real matrices. Let

$$||A||^2=\sum_{i,j=1}^{n}a_{ij}^2,$$

where (a_{ij}) are the entries of A, and let

$$[A,B]=AB-BA$$

be the commutator. Then the equation (2), in terms of matrices, can be formulated as follows

Conjecture 2. *For $n, m\geq 2$, we have*

$$(\sum_{r=1}^{m}||A_r||^2)^2\geq 2(\sum_{r<s}||[A_r,A_s]||^2).\tag{3}$$

Fixing n, m, we call the above inequality Conjecture $P(n,m)$.

Remark 1. For derivation of (2), see [6, Theorem 2]. Note that the prototype of the matrices are the traceless part of the second fundamental forms.

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2. Pinching theorems.

Let $A_1, \cdots, A_m$ be $n\times n$ symmetric matrices. Let $P(n,m)$ be the following conjecture [5, 4]:

Conjecture 3. *Using the above notations, we have*

$$2\sum_{i<j}||[A_i,A_j]||^2\leq\left(\sum_{i=1}^{m}||A_i||^2\right)^2.$$

In [1, 3], the following result was proved (cf. [1, pp 585, equation (5)]):

Theorem 2. *Using the same notations as above, we have*

$$2\sum_{i<j}||[A_i,A_j]||^2\leq\frac{3}{2}\left(\sum_{i=1}^{m}||A_i||^2\right)^2-\sum_{i=1}^{m}||A_i||^4.$$

□

We denote the above inequality to be $P'(n,m)$. In this note, we prove the following

Theorem 3.

$$P(n,m)\Rightarrow P'(n,m).$$

Proof. We assume that

$$||A_1||\geq\cdots\geq||A_m||.$$

We prove $P'(n,m)$ by induction: suppose $P'(n,m-1)$ is true. Then we have the following

Lemma 1. *If $P'(n,m)$ is true for*

$$||A_1||^2 \le \sum_{i=2}^m ||A_i||^2,$$

then $P'(n,m)$ is true for any $A_1, \cdots, A_m$.

Proof. We let $A_1 = tA_1'$ and assume that $||A_1'|| = 1$. Then inequality $P'(n,m)$ can be written as

$$\begin{aligned}&\frac{1}{2}t^4 - t^2\left(2\sum_{i=2}^m ||[A_1', A_i]||^2 - 3\sum_{i=2}^m ||A_i||^2\right)\\ &+\frac{3}{2}\left(\sum_{i=2}^m ||A_i||\right)^2 - \sum_{i=2}^m ||A_i||^4 - 2\sum_{2\le i<j} ||[A_i, A_j]||^2 \ge 0.\end{aligned} \tag{4}$$

By the inductive assumption, the total of the last three terms of the above is nonnegative. Let

$$a = 2\sum_{i=2}^m ||[A_1', A_i]||^2 - 3\sum_{i=2}^m ||A_i||^2. \tag{5}$$

If $a \le 0$, then then (4) is trivially true. On the other hand, if $a > 0$, then the minimum value is obtained at

$$t^2 = a.$$

Using the fact that $||[A_1', A_i]||^2 \le 2||A_i||^2$, we obtain:

$$||A_1||^2 \le \sum_{i=2}^m ||A_i||^2.$$

□

Proof of Theorem 3. If

$$||A_1||^2 \le \sum_{i=2}^m ||A_i||^2,$$

then

$$\left(\sum_{i=1}^m ||A_i||^2\right)^2 \le \frac{3}{2}\left(\sum_{i=1}^m ||A_i||^2\right)^2 - \sum_{i=1}^m ||A_i||^4.$$

Thus

$$P(n,m) \Rightarrow P'(n,m).$$

Since $P(3,m)$ is true by the main result in [4], can we get new pinching constant using this new inequality?

3. Proof of $P(n,3)$.

In this section, we prove the following

Theorem 4. *Let A, B, C be symmetric $n \times n$ matrices. Then*

$$(||A||^2 + ||B||^2 + ||C||^2)^2 \ge 2(||[A,B]||^2 + ||[B,C]||^2 + ||[C,A]||^2).$$

We first prove the following lemma:

Lemma 2. *Let $x \ge y \ge 0$. Let $(\eta_1, \cdots, \eta_n)$ be a unit vector. Then if $\{i,j\} \ne \{k,l\}$, we have*

$$(\eta_i - \eta_j)^2 x + (\eta_k - \eta_l)^2 y \le 2x + y.$$

Proof. If $i \notin \{k,l\}$ and $j \notin \{k,l\}$, then we have

$$(\eta_i-\eta_j)^2x+(\eta_k-\eta_l)^2y \leq 2(\eta_i^2+\eta_j^2)x+2(\eta_k^2+\eta_l^2)y \leq 2(\eta_i^2+\eta_j^2)x+2(1-\eta_i^2-\eta_j^2)y.$$

Thus we have

$$(\eta_i-\eta_j)^2x+(\eta_k-\eta_l)^2y \leq 2(x-y)+2y=2x\leq 2x+y.$$

On the other hand, if $i=k,l$ or $j=k,l$, then WLOG, we can assume that $i=k=1, j=2, l=3$. Thus we have

$$(\eta_1-\eta_2)^2x+(\eta_1-\eta_3)^2y=(\eta_1,\eta_2,\eta_3)\begin{pmatrix} x+y & -x & -y\\ -x & x & 0\\ -y & 0 & y\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}.$$

The largest eigenvalue of the above matrix is $x+y+\sqrt{x^2-xy+y^2}\leq 2x+y$. Since $\eta_1^2+\eta_2^2+\eta_3^2\leq 1$, we have

$$(\eta_1-\eta_2)^2x+(\eta_1-\eta_3)^2y\leq 2x+y,$$

as stated.

□

Lemma 3. *Suppose that $||A||^2+||B||^2+||C||^2=1$ and $||A||\geq||B||\geq||C||$. Let*

$$\lambda=\mathrm{Max}\,(||[A,B]||^2+||[B,C]||^2+||[C,A]||^2),$$

and let A,B,C be the maximum point. Then we have

$$2\lambda||A||^2=||[A,B]||^2+||[A,C]||^2.$$

Proof. Consider the function

$$F=||[A,B]||^2+||[B,C]||^2+||[C,A]||^2-\lambda'(||A||^2+||B||^2+||C||^2-1).$$

Using the Lagrange multiplier's method, for any symmetric matrix ξ, we have

$$\begin{aligned}\langle[A,B],[A,\xi]\rangle+\langle[C,B],[C,\xi]\rangle-\lambda'\langle B,\xi\rangle=0\\ \langle[B,A],[B,\xi]\rangle+\langle[C,A],[C,\xi]\rangle-\lambda'\langle A,\xi\rangle=0\\ \langle[B,C],[B,\xi]\rangle+\langle[A,C],[A,\xi]\rangle-\lambda'\langle C,\xi\rangle=0.\end{aligned}$$

Since ξ is arbitrary, we have

$$\begin{aligned}||[A,B]||^2+||[C,B]||^2-\lambda'||B||^2=0\\ ||[A,B]||^2+||[C,A]||^2-\lambda'||A||^2=0\\ ||[B,C]||^2+||[A,C]||^2-\lambda'||C||^2=0.\end{aligned}$$

Summing over the three equations, we have

$$2\lambda=\lambda'.$$

The lemma follows.

□

Proof of Theorem 4. Let

$$G=O(n)\times O(3).$$

The group acts on (A,B,C) as follows: let $Q\in O(n)$, then the Q action is

$$(A,B,C)\mapsto(QAQ^T,QBQ^T,QCQ^T);$$

let $Q_1=(q_{ij})\in O(3)$, then the Q_1 action is

$$(A,B,C)\mapsto(q_{11}A+q_{12}B+q_{13}C,\cdots,q_{31}A+q_{32}B+q_{33}C).$$

It is not hard to see that the inequality and the expression

$$||[A,B]||^2+||[B,C]||^2+||[C,A]||^2$$

are G invariant. Thus WLOG, we assume that A, B, C are orthogonal and consider the maximum of

$$||[A,B]||^2 + ||[A,C]||^2$$

under the constraint $||B||^2 = x, ||C||^2 = y$ and $x \geq y$. We assume that A is diagnolized. Let $A' = A/||A||$, and let

$$A' = \begin{pmatrix} \eta_1 & & \\ & \ddots & \\ & & \eta_n \end{pmatrix}$$

Then $\eta_1^2 + \cdots + \eta_n^2 = 1$.

Consider the function

$$g = \sum_{i,j} (\eta_i - \eta_j)^2 (b_{ij}^2 + c_{ij}^2) + \lambda_1 (\sum_{i,j} b_{ij}^2 - x) + \lambda_2 (\sum_{i,j} c_{ij}^2 - y) + \mu (\sum_{i,j} b_{ij} c_{ij}).$$

Using the Lagrange muliplier's method, at the maximum points, we have

$$2((\eta_i - \eta_j)^2 + \lambda_1) b_{ij} + \mu c_{ij} = 0$$
$$2((\eta_i - \eta_j)^2 + \lambda_2) c_{ij} + \mu b_{ij} = 0$$

for $i \geq j$.

WLOG, we assume that $(\eta_i - \eta_j)^2$ are different. If $\mu = 0$, then at most for one $i > j$ and one $k > l$, we have $b_{ij} \neq 0$ and $c_{kl} \neq 0$. Since B, C are orthogonal, if $b_{ij} \neq 0$ and $c_{kl} \neq 0$, then we have $(i,j) \neq (k,l)$. It follows that

$$\sum_{i,j} (\eta_i - \eta_j)^2 (b_{ij}^2 + c_{ij}^2) \leq (\eta_i - \eta_j)^2 x + (\eta_k - \eta_l)^2 y.$$

By Lemma 2, we have

$$\sum_{i,j} (\eta_i - \eta_j)^2 (b_{ij}^2 + c_{ij}^2) \leq 2x + y.$$

If $\mu \neq 0$, then we have

$$((2(\eta_i - \eta_j)^2 + \lambda_1)(2(\eta_i - \eta_j)^2 + \lambda_2) - \mu^2) b_{ij} c_{ij} = 0.$$

Thus at most two pairs of $b_{ij} c_{ij} \neq 0$ for $i > j$. On the other hand, since $\mu \neq 0$, $b_{ij} \neq 0$ iff $c_{ij} \neq 0$. There are several cases:

Case 1. Suppose that $b_{ij} c_{ij} \neq 0$ and $b_{kl} c_{kl} \neq 0$ for $\{i,j\} \neq \{k,l\}$. Then $b_{ii} = c_{ii} = 0$. The orthogonal condition implies that

$$b_{ij} c_{ij} + b_{kl} c_{kl} = 0.$$

Using the above conditions, we can assume that

$$b_{ij} = \sqrt{\frac{x}{2}} \cos\alpha, b_{kl} = \sqrt{\frac{x}{2}} \sin\alpha, c_{ij} = -\sqrt{\frac{y}{2}} \sin\alpha, c_{kl} = \sqrt{\frac{y}{2}} \cos\alpha.$$

Thus

$$\sum_{i,j} (\eta_i - \eta_j)^2 (b_{ij}^2 + c_{ij}^2) = (\eta_i - \eta_j)^2 (x \cos^2\alpha + y \sin^2\alpha) + (\eta_k - \eta_l)^2 (x \sin^2\alpha + y \cos^2\alpha).$$

Apparently, the maximum values are obtained at $\alpha = 0$ or $\frac{\pi}{2}$. Using Lemma 2, in either case, we have

$$\sum_{i,j} (\eta_i - \eta_j)^2 (b_{ij}^2 + c_{ij}^2) \leq 2x + y.$$

Case 2. If there is only one $b_{ij} \neq 0$, then we have

$$2b_{ij}c_{ij} + \sum_i b_{ii}c_{ii} = 0. \tag{6}$$

Thus we have

$$\sum_{i,j} (\eta_i - \eta_j)^2 (b_{ij}^2 + c_{ij}^2) \leq 4(b_{ij}^2 + c_{ij}^2).$$

An element computation gives that

$$4(b_{ij}^2 + c_{ij}^2) \leq 2x + y$$

using (6).

Case 3. If $b_{ij} = 0$ for $i > j$, then

$$\sum_{i,j} (\eta_i - \eta_j)^2 (b_{ij}^2 + c_{ij}^2) = 0 \leq 2x + y.$$

In summary, we have

$$||[A', B]||^2 + ||[A', C]||^2 \leq 2||B||^2 + ||C||^2.$$

Using Lemma 3, we have

$$2\lambda ||A||^2 \leq ||A||^2 (2x + y).$$

Since $||A|| \geq ||B|| \geq ||C||$, we have $2x + y \leq 1$. Thus $2\lambda \leq 1$. This is what we want to prove. □

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